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Efficient edge domination in regular graphs

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Abstract

An induced matching of a graph G is a matching having no two edges joined by an edge. An efficient edge dominating set of G is an induced matching M such that every other edge of G is adjacent to some edge in M. We relate maximum induced matchings and efficient edge dominating sets, showing that efficient edge dominating sets are maximum induced matchings, and that maximum induced matchings on regular graphs with efficient edge dominating sets are efficient edge dominating sets. A necessary condition for the existence of efficient edge dominating sets in terms of spectra of graphs is established. We also prove that, for arbitrary fixed $p \geq 3$, deciding on the existence of efficient edge dominating sets on p-regular graphs is NP-complete.

Keywords: Induced matchings; Domination in graphs; Regular graphs; Spectra of graphs; NP-completeness

1. Introduction

Let $G = (V, E)$ be a finite undirected simple graph. A *matching* of G is a set of mutually nonadjacent edges of G. An *induced matching* (IM) is a matching having no two edges joined by an edge. In other words, M is an induced matching (also called strong matching) of G if the subgraph of G induced by $V(M)$ is a 1-factor. A maximum induced matching is an induced matching of maximum cardinality.

Finding maximum IMs is NP-hard. This was proved independently for bipartite graphs by [17,1]. The same complexity has been proved for several classes of graphs including planar graphs [12], line graphs [13] and regular graphs [13,19,4]. On the other hand, there are polynomial time algorithms for the determination of maximum IMs for particular types of graphs such as chordal graphs [1], trapezoidal graphs [6] and cocomparability graphs [6]. See [2] for complexity results of determining maximum IMs on other classes of intersection graphs.

The concept of domination in graphs appears as a natural model for facility location problems, and has many applications in design and analysis of communication networks, network routing and coding theory, among others (see [10] and Section 9.2 in [9]).

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An edge of G *dominates* itself and every edge adjacent to it. An *efficient edge dominating set* (EEDS) of G is an induced matching that dominates every edge of G. Every edge of P_4 is an IM. Only the interior edge is an EEDS. Clearly not every graph has EEDSs. Cycle C_n has an EEDS (indeed three different EEDSs each of cardinality $\frac{n}{3}$) iff $n = 3k$, with $k \in \mathbb{N}$.

Deciding if a graph has an EEDS is NP-complete [8]. The result also holds for bipartite graphs [15], line graphs [8] and planar bipartite graphs [14]. For other classes of graphs such as bipartite permutation graphs [15] and chordal graphs [14] the problem is polynomially solvable.

In this paper we relate maximum IMs and EEDSs, showing that EEDSs are maximum IMs, and that maximum IMs on regular graphs with EEDSs are EEDSs. A necessary condition for the existence of EEDSs in terms of spectra of graphs is established. We also prove that deciding on the existence of EEDSs is NP-complete for p-regular graphs, for arbitrary fixed $p \geq 3$. As an immediate consequence we obtain the NP-completeness of recognizing (vertex) efficient domination (also called perfect codes) on $2p$-regular (line) graphs, for fixed $p \geq 2$. An *efficient dominating set* of G is an independent (no two elements are adjacent) set of vertices S such that every vertex in $V \setminus S$ is adjacent to exactly one element in S. The NP-completeness of recognizing efficient dominating sets on cubic graphs was proved by [11] and, to our knowledge, this was the only result on the complexity of efficient domination on regular graphs.

2. Basic results and generalities

Every EEDS is a maximal IM. The following theorem states that all EEDSs have the same size which is the size of a maximum IM.

Theorem 2.1. *If M is an EEDS of a graph G, M is a maximum IM of G.*

Proof. Let T be a matching of G, with $|T| > |M|$. Consider $M' = M \setminus T$ and $T' = T \setminus M$. Note that $|T'| > |M'| \geq 1$, and M' dominates every edge of T'. Hence, there is an edge in M' that dominates two different edges of T', showing that T' (and, consequently, T) is not an IM. $\square$

The next result gives a lower bound on the size of the EEDSs by means of the spectrum of the adjacency matrix A_G of graph G.

If e is an edge of G let E_e be the set of edges of G that e dominates, and $G_e = (V, E_e)$ the spanning subgraph of G with edge set E_e. If u is a vertex of G let E_u be the set of edges incident in u. Denote by $\sigma^+(A)$ and $\sigma^-(A)$ the number of positive and negative eigenvalues of matrix A, respectively.

Theorem 2.2. *If M is an EEDS of a graph G, $|M| \geq \frac{1}{2} \max\{\sigma^+(A_G), \sigma^-(A_G)\}$.*

Proof. Note that

(a) for every edge $e = [u, v]$ of G, $A_{G_e} = A_u + A_{v\setminus e}$, where A_u and $A_{v\setminus e}$ are the adjacency matrices of the spanning subgraphs of G: $S_u = (V, E_u)$ and $S_{v\setminus e} = (V, E_v \setminus \{e\})$, respectively, and
(b) if M is an EEDS, $A_G = \sum_{e\in M} A_{G_e}$.

Since S_u and $S_{v\setminus e}$ are stars ($S_{v\setminus e}$ with possibly no edges), $\sigma^+(A_u) = \sigma^-(A_u) = 1$ and $\sigma^+(A_{v\setminus e}) = \sigma^-(A_{v\setminus e}) \leq 1$, and the result follows from remarks (a) and (b) and the fact (Corollary 1.4 in [7]) that for symmetric matrices A and B, $\sigma^+(A + B) \leq \sigma^+(A) + \sigma^+(B)$ and $\sigma^-(A + B) \leq \sigma^-(A) + \sigma^-(B)$. $\square$

Clearly, if M is an IM of G,

$$|M| \leq \frac{|E(G)|}{2\delta(G) - 1}, \tag{1}$$

where $\delta(G)$ denotes the minimum degree of the vertices of G.

Combining Theorem 2.2 with (1) may be useful to assess that EEDSs do not exist. For instance this allows to conclude, that no EEDS can be found in the line graph of $K_{n,n}$, with $n \geq 4$, and in the graph resulting from deleting a perfect matching from the line graph of $K_{n,n}$, with $n \geq 6$ even. Another example is the graph depicted in Fig. 1.

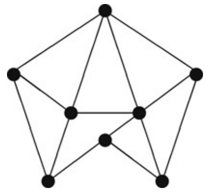

Fig. 1. A graph G such that $\frac{1}{2}\max\{\sigma^+(A_G), \sigma^-(A_G)\} = \frac{5}{2}$ and $\frac{|E(G)|}{2\delta(G)-1} = \frac{14}{5}$.

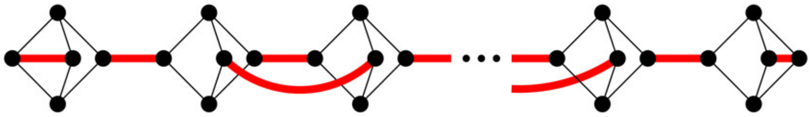

Fig. 2. A cubic graph of order $10k$ with an EEDS.

3. EEDSs in regular graphs

The cardinality of an EEDS M of a p-regular graph equals

$$|M| = \frac{|E(G)|}{2p-1}.$$

The 2-regular connected graphs with EEDSs are the cycles C_n with $n = 3k$, for $k \in \mathbb{N}$. However, cubic graphs with EEDSs are asymptotically rare. This follows directly from a result of [4] which proves that the sizes of the IMs of cubic graphs of order n do not asymptotically almost surely exceed $0.282069\,n$. Despite this fact, there are cubic graphs of order $|V(G)| = n$ which have EEDSs, whenever $\frac{|E(G)|}{5} = 0.3\,n \in \mathbb{N}$ (see Fig. 2).

In general, graphs with EEDSs have maximum IMs which are not EEDSs. Nevertheless, for regular graphs with EEDSs the converse of Theorem 2.1 holds.

Theorem 3.1. *Every maximum IM of a p-regular graph with an EEDS is an EEDS.*

Proof. Let $M \subseteq E(G)$ be an IM of a p-regular graph G. The family of edge subsets $\{E_e, e \in M\}$ is pairwise disjoint and thus $|\bigcup_{e\in M} E_e| = \sum_{e\in M} |E_e| = (2p-1)|M| \le |E(G)|$. Clearly $(2p-1)|M| = |E(G)|$ iff M dominates every edge in G. □

We now derive a necessary condition for the existence of EEDSs in a p-regular graph $G = (V, E)$.

Consider variables x_e associated to every edge e of G. The EEDSs are the 0–1 solutions of the linear system of equations

$$\sum_{a\in E_e} x_a = 1, \quad \text{for every } e \in E. \tag{2}$$

Since G is p-regular, the vector $\bar{x} \in \mathbb{R}^{|E(G)|}$ defined by $\bar{x}_e = \frac{1}{2p-1}$, for every edge e of G, is a solution of (2). Let P_G be the matrix of the coefficients of the left-hand side of (2). If G has an EEDS and $p > 1$, then P_G is singular. This observation will be used to prove the following necessary condition for the existence of EEDSs on p-regular graphs.

Theorem 3.2. *If a p-regular graph G, with $p > 1$, has an EEDS then -1 is an eigenvalue of the adjacency matrix $A_{L(G)}$ of its line graph $L(G)$.*

Proof. The above observation asserts that 0 is an eigenvalue of P_G. To conclude the proof note that $P_G = A_{L(G)} + I$, where I is the identity matrix of order $|E(G)|$. □

Since $\lambda \neq -2$ is an eigenvalue of $A_{L(G)}$ iff $\lambda + 2 - p$ is eigenvalue of A_G (Theorem 2.15 in [3]), Theorem 3.2 can be restated as follows.

Theorem 3.3. *If a p-regular graph G, with $p > 1$, has an EEDS then $1 - p$ is an eigenvalue of the adjacency matrix A_G of G.*

It should be noted that the above results could also be derived from Theorem 3.3 of [18].

Theorem 3.3 (or 3.2) allow us to exclude from having EEDSs eight graphs (graphs labelled C11, C12, C13, C14, C16, C18, C23 and C24 from the Atlas of Graphs [16] among the 19 connected cubic graphs of order 10. Indeed, -2 is missing from the spectra of the adjacency matrices of these graphs.

We proceed proving that recognizing regular graphs with EEDSs is NP-complete.

4. Complexity

We use the NP-completeness of a version of the one-in-three 3-satisfiability problem (problem [LO4] in [5]) to establish that, for each $p \geq 3$, checking which p-regular graphs have EEDSs is NP-complete. (We follow the terminology from the description of the one-in-three 3-satisfiability problem given in [5]). An instance of this problem (which will be denoted by 1-in-3SAT) consists of a set X of boolean variables and a collection $C = \{c_1, \ldots, c_m\}$ of m clauses over X, such that each c_i has $|c_i| = 3$ and does not contain any negated literal. One-in-3SAT asks whether there is a truth assignment for X under which each clause in C has exactly one true literal.

Consider an instance of 1-in-3SAT and $3 \leq p \in \mathbb{N}$. Let m_x be the number of occurrences of literal x in the collection of clauses C, and assume that $m_x \geq 2$ and $m = (p-1)k$, with $k \in \mathbb{N}$. No loss of generality follows from these assumptions since copies of clauses can be added to C to achieve both requirements. (No more than $m + p - 2$ copies are necessary.) From that instance we will construct a p-regular graph G_p. We begin with the following construction.

Each clause $c = (x\ y\ z)$ gives rise to the collections:

G_c of $p-2$ isolated vertices, and
G'_c of $p-1$ vertex disjoint triangles, each one with vertices labelled x, y and z.

Each variable x gives rise to the collections:

G_x of $m_x(p-2)$ isolated vertices, and
G'_x of $m_x(p-2)$ vertex disjoint paths P_3.

For each vertex v of G_x choose a clause c which contains the literal x and connect v with some vertex in G_c. Let the connections be made so that, for every clause c, there will be exactly three edges incident to each vertex of G_c.

Let u and v denote the end vertices of each path P_3 of G'_x. Each vertex u will be connected to $p-1$ triangle vertices with labels x, in such way that each triangle vertex has degree p (counting the edges of the triangle). The vertices v will be connected to the vertices of G_x in order to obtain a connected $(p-1)$-regular bipartite graph, which we call H_x.

In what follows we will refer the edges of each P_3 incident to v and u as the f-edge and the t-edge, respectively.

Let this construction be denoted by $\widetilde{G}_p$.

If $\widetilde{G}_p$ has an EEDS M the following holds:

(a) M includes either a t-edge or an f-edge from each P_3 in G'_x. This immediately follows since every P_3 of G'_x is connected to a triangle of G'_c, and every triangle has an edge in M.
(b) M includes either all t-edges or else all f-edges from each G'_x. This is a consequence of H_x being a connected bipartite graph which, according to (a), has no edges in M.
(c) If the t-edges of G'_x are in M, all edges connecting vertices of G_x with vertices in G_c are in M, for every clause c where literal x occurs. Otherwise M would not dominate some of these edges.
(d) $V(M)$ includes all vertices of G_c. Otherwise M would not dominate any of the three edges incident to a vertex of G_c.

Let us identify each truth assignment T for X with a set E_T of t and f-edges. If variable x is true under T all t-edges (and no f-edge) of G'_x are in E_T. Conversely, if variable x is false under T all f-edges (and no t-edge) of G'_x are in E_T.

We are in a condition to prove the following result.

Lemma 4.1. *Let X (set of variables) and C (collection of m clauses) be an instance of 1-in-3SAT and $3 \leq p \in \mathbb{N}$, such that $m_x \geq 2$, for every x in X and $m = (p-1)k$, with $k \in \mathbb{N}$. There is a truth assignment for X such that each clause in C has exactly one true literal iff the corresponding graph $\widetilde{G}_p$ has an EEDS.*

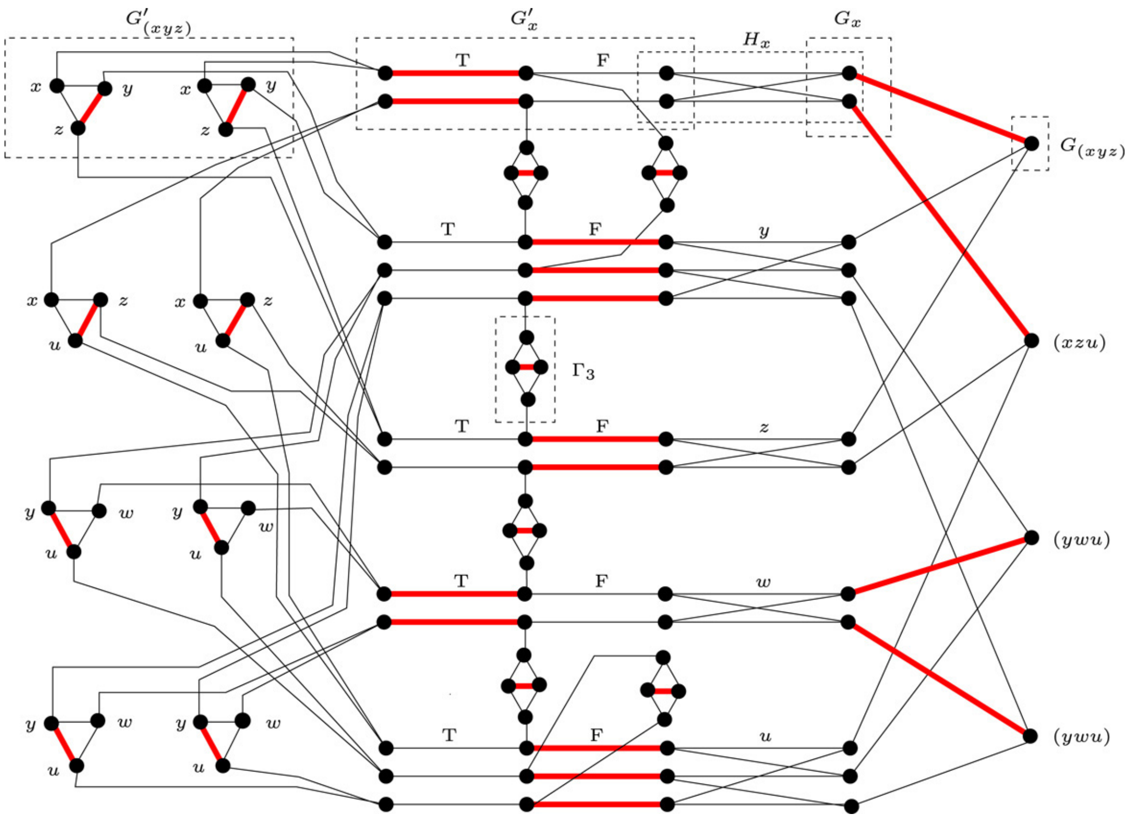

Fig. 3. The cubic graph G_3 corresponding to the collection of clauses $C = \{(\mathbf{x}yz), (\mathbf{x}zu), (y\mathbf{w}u)^2\}$. The thick edges identify the EEDS corresponding to the satisfying truth assignment under which variables x and w are true and y, z and u are false.

Proof. Assume that T is a truth assignment for X such that each clause in C has exactly one true literal. Let M_T be obtained by enlarging E_T with the following edges. If variable x is true under T and c is a clause where literal x occurs, add to M_T all edges with a vertex in G_x and the other vertex in G_c. Add also to M_T the edge opposite to the vertex labelled x in every triangle of G'_c. No further edges are added. Since under T each clause has exactly one true literal, M_T is clearly an EEDS.

The remarks (a)–(d) clearly imply that every EEDS corresponds to a truth assignment where each clause has exactly one true literal. □

We now define a p-regular graph G_p by adding some vertices and edges to $\widetilde{G}_p$ in such a way that M is an EEDS of G_p iff $M \cap E(\widetilde{G}_p)$ is an EEDS of $\widetilde{G}_p$.

Let Γ_p be the graph consisting of $p-1$ triangles with a common edge. Let $k \in \mathbb{N}$ such that $m = (p-1)k$. If $p = 3$ [resp. $p > 3$] take $3k$ [resp. $4k$] copies of Γ_p. Connect the $(p-1)$ vertices with degree two of each copy of Γ_p to the vertices of $\widetilde{G}_p$ with degree less than p in such a way that in the resulting graph G_p every vertex has degree p. This is possible since the number of vertices in $\widetilde{G}_p$ with degree two equals $3(p-2)m$ and the number of vertices with degree three equals $(p-2)m$.

Fig. 3 represents the final graph G_3 for $C = \{(xyz), (xzu), (ywu)^2\}$.

Graph G_p shall be used to prove main result of this section.

Theorem 4.2. *For each $p \in \mathbb{N}$, $p \geq 3$, recognizing if a p-regular graph has an EEDS is NP-complete.*

Proof. Since the common edge of the triangles of each Γ_p has to belong to every EEDS of G_p, M is a EEDS of G_p iff $M \cap E(\widetilde{G}_p)$ is an EEDS of $\widetilde{G}_p$.

Moreover, since deciding whether a graph has an EEDS is clearly in NP and the size of the p-regular graph G_p is polynomial in the size of the 1-in-3-SAT instance the result follows. □

As an immediate consequence of Theorem 4.2 we obtain the following result which extends the NP-completeness of deciding on the existence of efficient dominating sets on cubic graphs [11].

Corollary 4.3. *For each* $3 < p = 2k \in \mathbb{N}$, *recognizing if a* p*-regular graph has a (vertex) efficient dominating set is NP-complete.*

Proof. The EEDSs of the p-regular graph G_p are the efficient dominating sets of its line graph which is $2(p-1)$-regular. □

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